Three–loop QED Vacuum Polarization and the Four–loop Muon Anomalous Magnetic Moment[*][†]

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Three–loop contributions to massive QED vacuum polarization are evaluated by a combination of analytical and numerical techniques. The first three Taylor coefficients, at small q^2, are obtained analytically, using d–dimensional recurrence relations. Combining these with analytical input at threshold, and at large q^2, an accurate Padé approximation is obtained, for all q^2. Inserting this in the one–loop diagram for the muon anomalous magnetic moment, we find reasonable agreement with four–loop, single–electron–loop, muon–anomaly contributions, recently re–evaluated by Kinoshita, using 8–dimensional Monte–Carlo integration. We believe that our new method is at least two orders of magnitude more accurate than the Monte–Carlo approach, whose uncertainties appear to have been underestimated, by a factor of 6.

1. Introduction

We describe a method, previously tested[1] in two–loop QCD, to approximate, to high accuracy, three–loop contributions to QED vacuum polarization, using new analytical results for the small momentum–transfer limit, combined with asymptotic[2,3,4] and threshold[5] results. Related contributions to the four–loop muon anomalous magnetic moment are computed, to test an evaluation[6] that was undertaken in response to a previous discrepancy between numerical[7] and analytical[8] work.

In the on–shell (OS) renormalization scheme of conventional QED, the renormalized photon propagator has a denominator $(1 + \Pi(z))$, where $z \equiv q^2/4m^2$, with an electron mass m, and the vacuum polarization function, $\Pi(z)$, vanishes at $z = 0$. Non–relativistic consideration of the electron–positron system yields information about the threshold[5] behaviour, as $z \to 1$. Moreover, the $\overline{\rm MS}$ asymptotic behaviour[3,4], as $z \to -\infty$, combined with relations[8,9] between the $\overline{\rm MS}$ and OS

schemes, yields two terms of the asymptotic expansion in powers of $1/z$.

The crucial new ingredient is our use of recurrence relations[9], to obtain the first three terms of the expansion as $z \to 0$. Combining 6 analytical data with Padé[1,10] (or hypergeometric[1,11]) approximations, we shall produce reliable fits, for all z, and hence check four–loop muon–anomaly contributions[6].

2. Small–momentum behaviour

We evaluated, to 3 loops, the first 3 moments in the $z \to 0$ expansion

$$\Pi(z) = \sum_{n>0} C_n \, z^n + \mathrm{O}(\alpha^4) \,,$$

by intensive application of d–dimensional recurrence relations to three–loop massive vacuum diagrams[9], with propagators raised to powers up to 11, since up to 8 differentiations w.r.t. the external momentum q are required before setting it to zero. This put great demands on the REDUCE package RECURSOR[9], which used 80 MB of memory, for 2 days, on a DecAlpha machine, after hand–tuning the procedures, to minimize recomputation of integrals, and to allow safe truncation in $\varepsilon = (4-d)/2$. The gauge invariance of C_1 and C_2 was verified for all ε. After OS mass[12,13] and charge[9] renormalization, we obtained the finite $\varepsilon \to 0$ limits

$$
\begin{aligned}
C_1 &= \left\{ N^2 \left[\tfrac{8}{15}\zeta_2 + \tfrac{203}{864}\zeta_3 - \tfrac{11407}{11664} \right] \right. \\
&\quad \left. + N \left[\left(1 - \tfrac{8}{5}\ln 2\right)\zeta_2 + \tfrac{22781}{6912}\zeta_3 - \tfrac{8687}{3456} \right] \right\} \frac{\alpha^3}{\pi^3} \\
&\quad + \tfrac{82}{81}\, N \frac{\alpha^2}{\pi^2} + \tfrac{4}{15}\, N \frac{\alpha}{\pi} \,, \\
C_2 &= \left\{ N^2 \left[\tfrac{16}{35}\zeta_2 + \tfrac{14203}{73728}\zeta_3 - \tfrac{1520789}{1658880} \right] \right. \\
&\quad \left. + N \left[\tfrac{6}{7}\left(1 - \tfrac{8}{5}\ln 2\right)\zeta_2 + \tfrac{4857587}{184320}\zeta_3 - \tfrac{223404289}{7464960} \right] \right\} \frac{\alpha^3}{\pi^3} \\
&\quad + \tfrac{449}{675}\, N \frac{\alpha^2}{\pi^2} + \tfrac{4}{35}\, N \frac{\alpha}{\pi} \,, \\
C_3 &= \left\{ N^2 \left[\tfrac{128}{315}\zeta_2 + \tfrac{12355}{55296}\zeta_3 - \tfrac{83936527}{93312000} \right) \right] \\
&\quad \left. + N \left[\tfrac{16}{21}\left(1 - \tfrac{8}{5}\ln 2\right)\zeta_2 + \tfrac{33067024499}{206438400}\zeta_3 - \tfrac{885937890461}{4644864000} \right] \right\} \frac{\alpha^3}{\pi^3} \\
&\quad + \tfrac{249916}{496125}\, N \frac{\alpha^2}{\pi^2} + \tfrac{64}{945}\, N \frac{\alpha}{\pi} \,,
\end{aligned}
$$

(1)

(2)

(3)

where we follow common practice[8], by allowing for N degenerate leptons. In pure QED, $N = 1$; formally, the powers of N serve to count the number of electron loops. Our principal interest, for consideration of four–loop muon–anomaly contributions[6], is $\Pi_3^{[1]}\alpha^3/\pi^3$, the three–loop contributions to Π that involve a *single* electron loop. The moments of $\Pi_3^{[1]}(z)$ are given by the coefficients of $N\alpha^3/\pi^3$. (We shall not need the $N^2\alpha^3/\pi^3$ terms in Section 6, since the muon–anomaly contributions of the diagrams with two electron loops are better understood[8,14].)

3. Large–momentum behaviour

The situation regarding the $z \to -\infty$ behaviour of $\Pi(z)$ was unclear, until recently, because three–loop $\overline{\text{MS}}$ QCD results[2] had been altered, while obtaining QED results[3], in the belief (now known[4] to be mistaken) that the former contained errors. Further calculation[15] confirmed the QCD results[2] and hence invalidated the $\text{O}(1/z)$ QED results[3]. Accordingly, we thought it prudent to derive the OS asymptotic behaviour ourselves, from first principles, using the REDUCE package SLICER, which had been written specifically to check[8] the leading, massless, $\overline{\text{MS}}$ behaviour, obtained[3] with the SCHOONSCHIP package MINCER[16].

In our *ab initio* derivation of the asymptotic OS result for $\Pi_3^{[1]}$, we used neither the $\overline{\text{MS}}$ scheme, nor MINCER. Instead, the asymptotic expansion of the bare diagrams was obtained, in d dimensions, using SLICER, and the bare charge and mass were transformed directly to the physical charge and mass, using multiplicative OS renormalizations[9,13], obtained by RECURSOR. Setting $\varepsilon = 0$, we obtained a finite OS result of the form

$$\Pi_3^{[1]}(z) = A(z) + B(z)/z + \text{O}(L^3/z^2) \,,$$

where $L \equiv \ln(-4z) = \ln(-q^2/m^2)$ and

$$
\begin{aligned}
A(z) &= -\tfrac{121}{192} + \tfrac{5}{2}\zeta_5 - \tfrac{99}{64}\zeta_3 + 2\left(\ln 2 - \tfrac{5}{8}\right)\zeta_2 + \tfrac{1}{32}L \,, & (4) \\
B(z) &= \tfrac{139}{48} - \tfrac{35}{24}\zeta_5 - \tfrac{41}{48}\zeta_3 + 3\left(\ln 2 - \tfrac{5}{8}\right)\zeta_2 - \tfrac{3}{32}\left(L - 6L^2\right) \,. & (5)
\end{aligned}
$$

Using finite transformations[9,12] from physical to $\overline{\text{MS}}$–renormalized quantities, one obtains, from our OS result, an $\overline{\text{MS}}$ asymptotic behaviour identical to that which would[17] have been obtained from the QCD analysis[2], had it not been miscorrected in the course of deriving QED results[3]. As a result of our, and other[15], work, a (second) erratum[4] to the QED work[3] was issued.

In conclusion, we are confident of our OS QED result, since it is quite independent of previous works[2,3,15] and, eventually[4,17], in agreement with them.

4. Threshold behaviour

The leading threshold behaviour, at 3 loops, is determined by non–relativistic quantum mechanics: $\Pi_3^{[1]}(z) = \tfrac{1}{24}\pi^5(1 - z)^{-1/2} + \text{O}(\ln(1 - z))$, as $z \to 1$. Moreover, it appears[5,18] that a stronger statement can be made, namely that the first relativistic correction to the spectral density, $\rho(t) \equiv \text{Im}\,\Pi(t + \text{i}0)/\pi$, at any given order in α, is cancelled in the combination $(1 + 4\alpha/\pi)\rho(t)$. At the two–loop level, the exact relativistic results confirm that $\rho_2(t) + 4\,\rho_1(t) = \pi^2 + \text{O}(v^2)$ is free of a term of first order in $v \equiv (1 - 1/t)^{1/2}$. The corresponding cancellation at 3 loops is expected[18] to occur in $v(\rho_3^{[1]} + 4\,\rho_2) = \tfrac{1}{24}\pi^4 + \text{O}(v^2)$, implying that

$$\lim_{z \to 1}\left(\Pi_3^{[1]}(z) + 4\,\Pi_2(z) - \frac{\pi^5}{24(1 - z)^{1/2}}\right) = \text{constant} \,, \qquad (6)$$

with an unknown value for the constant, but no logarithmic singularity.

5. Approximation method

We express the analytical results (1–6) as properties of the combination

$$\tilde{\Pi}_3^{[1]}(z) \equiv \Pi_3^{[1]}(z) + 4\,\Pi_2(z) + (1-z)\,G(z)\left(\frac{9}{4}G(z) + \frac{31}{16} + \frac{229}{32z}\right) - \frac{229}{32z} - \frac{173}{96}, \quad (7)$$

where $G(z) \equiv {}_2F_1\left(1,1;\frac{3}{2};z\right)$ is given by $(z^2 - z)^{-1/2}\mathrm{arcsinh}(-z)^{1/2}$, on the negative real axis, and the two–loop term, $\Pi_2(z)$, is quadratic in $G(z)$ and involves a trilogarithm[8] and its derivative.

The data are conveniently encoded by the moments

$$M(n) \equiv \int_1^\infty \frac{\mathrm{d}t}{t^{n+1}}\,\tilde{\rho}_3^{[1]}(t)$$

of the spectral density of $\tilde{\Pi}_3^{[1]}$. At small z, we have $\tilde{\Pi}_3^{[1]}(z) = \sum_{n>0} M(n)\,z^n$ and hence obtain $M(1)$, $M(2)$, $M(3)$ from the coefficients of $N\alpha^3/\pi^3$ in the results of Eqs (1,2,3) for the moments of $\Pi(z)$, after taking account of the known moments of the additional terms in Eq (7). At large z, the logarithmic singularities of these additional terms cancel, by deliberate construction, those of Eqs (4,5), whose constant terms therefore determine $M(0)$ and $M(-1)$, respectively. Finally, the threshold Coulomb singularity of Eq (6) gives the large–n behaviour of the ratio

$$R(n) \equiv \frac{M(n)}{C(n)} = \frac{\pi^4}{24} + \mathrm{O}(1/n), \qquad C(n) \equiv \int_1^\infty \frac{\mathrm{d}t}{t^{n+1}}\,\rho_3^{[c]}(t) = \frac{(1)_{n+1}}{\left(\frac{1}{2}\right)_{n+2}},$$

where $C(n)$ is the moment of a spectral density $\rho_3^{[c]}(t) \equiv t^{-3/2}(t-1)^{-1/2}$, with a coulombic $1/v$ threshold singularity and the same convergence properties, at large t, as $\tilde{\rho}_3^{[1]}(t)$. Note that a further datum, namely the absence of a logarithmic singularity in Eq (6), corresponds to the absence of an $\mathrm{O}(1/n^{1/2})$ term in $R(n)$, as $n \to \infty$, partly accounting for the remarkable uniformity of our final analytical database:

$$
\begin{array}{lll}
R(-1) = & -\frac{3}{2}\left(\ln 2 - \frac{5}{8}\right)\zeta_2 + \frac{41}{96}\zeta_3 + \frac{35}{48}\zeta_5 + \frac{911}{384} & = 3.473\,721\,028\,889 \\[4pt]
R(0) = & -\frac{3}{2}\left(\ln 2 - \frac{5}{8}\right)\zeta_2 + \frac{1065}{256}\zeta_3 - \frac{15}{8}\zeta_5 + \frac{307}{256} & = 4.087\,577\,319\,347 \\[4pt]
R(1) = & -\frac{3}{2}\left(\ln 2 - \frac{5}{8}\right)\zeta_2 + \frac{113905}{36864}\zeta_3 + \frac{35551}{55296} & = 4.188\,975\,919\,282 \\[4pt]
R(2) = & -\frac{3}{2}\left(\ln 2 - \frac{5}{8}\right)\zeta_2 + \frac{34003109}{1179648}\zeta_3 - \frac{7229411363}{238878720} & = 4.216\,954\,083\,059 \\[4pt]
R(3) = & -\frac{3}{2}\left(\ln 2 - \frac{5}{8}\right)\zeta_2 + \frac{33067024499}{167772160}\zeta_3 - \frac{6144308789323}{26424115200} & = 4.224\,481\,581\,719 \\[4pt]
R(\infty) = & \frac{15}{4}\zeta_4 & = 4.058\,712\,126\,417
\end{array}
$$

which has been obtained from 3 quite disparate regimes.

Now we map[1] the cut z–plane onto the unit disk, and define a mapped function

$$P(\omega) \equiv \frac{1-\omega}{(1+\omega)^2}\left(\tilde{\Pi}_3^{[1]}(z) - \tilde{\Pi}_3^{[1]}(\infty)\right), \qquad z = \frac{4\omega}{(1+\omega)^2}, \qquad (8)$$

which is analytic for $|\omega| < 1$, with the cut mapped to the unit circle. The 6 data then determine $\{P(-1), P(0), P'(0), P''(0), P'''(0), P(1)\}$, allowing us to construct [2/3] and [3/2] Padé approximants[1], with benign poles outside the unit disk, and imaginary parts on the unit circle that accurately approximate the spectral density. The differences between these two approximations are very small, for all $|\omega| \le 1$.

6. Four–loop contribution to the Muon Anomalous Magnetic Moment

Our simple rational approximations to $P(\omega)$ reproduce, exactly, all known data on $\Pi_3^{[1]}$, as well as its analyticity structure. We now use them to calculate the four–loop contribution[6], $a_\mu = A_4^{[1]}\alpha^4/\pi^4$, to the muon anomaly, $(g/2-1)_\mu$, due to insertion of three–loop, single–electron–loop vacuum polarization diagrams into the one–loop anomaly diagram. A typical diagram is
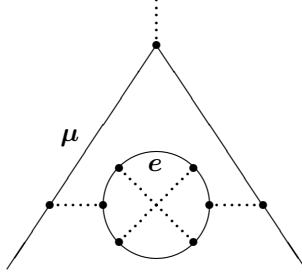

The resulting coefficient of α^4/π^4 in the muon anomaly is given by[14]

$$A_4^{[1]} = -\int_0^1 dy\,(1-y)\,\Pi_3^{[1]}(z)\,, \qquad z = -\frac{m_\mu^2}{4m^2}\frac{y^2}{1-y}\,. \tag{9}$$

We calculate the integral using [3/2], [2/3], and [2/2] Padé approximants to $P(\omega)$. In the [2/2] approximants we omit a piece of data from each regime, obtaining

Input	all	all	omit Eq (3)	omit Eq (5)	omit Eq (6)
Padé	[3/2]	[2/3]	[2, 2]	[2/2]	[2/2]
$-A_4^{[1]}$	0.230 362 20	0.230 362 18	0.230 360 42	0.230 366 94	0.230 361 49

with a muon mass $m_\mu = 206.768\,262\,m$. The stability is remarkable: changing the Padé method from [3/2] to [2/3] changes the output by 1 part in 10^7; removing a piece of data, from any of the 3 regimes, changes it by no more than 2 parts in 10^5. The improvement from using 6 inputs, as opposed to 5, is greatest in the case of including the asymptotic result of Eq (5). In contrast, the Coulomb datum, $R(\infty) = \frac{1}{24}\pi^4$, improves the result by only 3 parts in 10^6, since the muon–anomaly integral (9) involves only space–like momenta. The smallness of our spread of results demonstrates a high degree of consistency in the input, making the possibility of analytical error very remote. Being conservative, we take the range of [2/2] results as a measure of our uncertainty, and arrive at $A_4^{[1]} = -0.230362(5)$, to be compared with a recent[6] Monte–Carlo result, $A_4^{[1]} = -0.2415(19)$, obtained using VEGAS, in preference to RIWIAD (which gave[7] a grossly discrepant value, amended[6] in the light of a renormalization–group analysis[8]). In visual terms, the comparison is
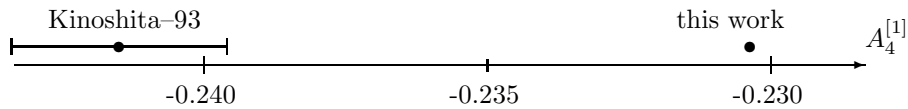
To verify that this discrepancy is not an artifact of the Padé method, we also tried a hypergeometric[1,11] method, i.e. a polynomial fit to $\tilde{\rho}_3^{[1]}(t)/\rho_3^{[c]}(t) - \frac{1}{24}\pi^4 = \sum_{k>1} c_k v^k$, with 5 coefficients, fixed by $R(n) - \frac{1}{24}\pi^4 = \sum_k c_k \left(\frac{1}{2}\right)_{n+2} / \left(\frac{k+1}{2}\right)_{n+2}$. As might be expected, the resultant fit to the spectral density $\rho_3^{[1]}(t)$ was less smooth than in our Padé methods. Nevertheless, the shift in the value for $A_4^{[1]}$ was two orders of magnitude less than the disagreement with the Monte–Carlo result.

In conclusion, we stress that the analytical data of Eqs (1–6) exhibit a high degree of internal consistency, making it most unlikely that any of them is in error. Padé approximants for the mapping (8) of the well–behaved function (7) enable us to evaluate the muon–anomaly contribution (9) with an uncertainty of 2 parts in 10^5. Our result is in reasonable agreement with a recent, lower–precision, Monte–Carlo re–evaluation[6], whose uncertainties appear to have been underestimated by a factor of 6, which is a great improvement on the situation revealed by a previous discrepancy between analytical[8] and numerical[7] work.

Acknowledgements

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